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Continued Fractions

A Brief Overview

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Mathematics

Abstract

We study continued fractions as mathematical representations of real numbers. We describe the related procedures geometrically and algebraically, including their relationships with the Euclidean algorithm for finding the greatest common divisor. A summary of the history of the apparatus is given from ancient Babylon to the 19th century, as well as applications from the fields of number theory, knot theory, applied mathematics, and engineering. We give a description of computations on continued fractions and the taking of reciprocals. We define the convergents of a continued fraction as certain rational numbers and study their properties. We prove the uniqueness of the continued fraction representation of any rational or real number. Several commonly used irrational numbers are represented as continued fractions along with proofs. We provide two computer programs written in Python in order to evaluate the decimal representation and the convergents of a given finite simple continued fraction.

Description

An expression of the following form:

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cfrac{b_4}{a_5 + \cdots}}}}}$$

is called a continued fracti… les, which can be real or complex numbers, functions, or other mathematical constructs. The continued fraction can terminate after some finite number of i,j, or continue indefinitely.

A simple continued fraction is a similar expression which only contains ones (1's) in the numerators.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

This expression can be repre… $a_0, a_1, a_2, \ldots, a_n, \ldots$, and we can use the following notation:

$$[a_0, a_1, a_2, \ldots, a_n, \ldots].$$

The simple continued fraction constructed from the first m terms is known as the mth convergent of the number.

Results

- Every rational and real number has a unique representation as a simple continued fraction.
- The partial convergents of a simple continued fraction give us the best possible rational approximations for any real number.
- Any written decimal expansion of a real number is also an approximation, and therefore the simple continued fraction representation of a real number is a more efficient approximation.
- Because continued fractions involve recursion, computers are a natural tool to help compute their values and the values of their rational convergents.

Examples

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}}$$

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}}}}$$

The golden ratio: $$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}$$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{\cdots}}}}}$$

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