Infinite Sets

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Cardinality

Cardinality is transitive even for the infinite sets. If set A and set B have the same cardinality, then there is a one-to-one correspondence from set A and set B.

Example: D= $\mathbb{Z}$ and E= {...-4,-2,0,2,4,...}
The function f: $\mathbb{Z} \rightarrow$ E , given by $f(n)=2n$ is a 1:1 correspondence. So even though E$\subset\mathbb{Z}$, |E|=|$\mathbb{Z}$ |

D	1:1	E
0	$\leftarrow - - \rightarrow$	0
1	$\leftarrow - - \rightarrow$	2
-1	$\leftarrow - - \rightarrow$	-2
2	$\leftarrow - - \rightarrow$	4
-2	$\leftarrow - - \rightarrow$	-4
.		.
.		.
.		.
n	$\leftarrow - - \rightarrow$	$2n$

1:1 Correspondence

1:1 Correspondence between two sets (call them A and B) is an assignment in which pairs each element of A with one and only one element of B in such a way that each element of B is paired with one and only one element of A.

Example: A={a,b,c,d} and B={x,y,z,w}

A	1:1 Correspondence	B
a	$\leftarrow - - - - \rightarrow$	w
b	$\leftarrow - - - - \rightarrow$	x
c	$\leftarrow - - - - \rightarrow$	y
d	$\leftarrow - - - - \rightarrow$	z

Infinite Set:

An Infinite set is non empty countable (or denumerable) if it has the cardinality equal to *aleph null*- the cardinality, the set of all natural numbers **N**.

Example: **N**= {1,2,3,4,5...} and **A**= $\{a_1, a_2, a_3, a_4, \ldots\}$

Note: If **A** countable, we can make a list of elements of **A**, i.e find a 1:1 correspondence with the set of natural numbers **N**.

$$\begin{array}{ccc} \mathrm{N} & & A \\ 1 & \leftarrow -- \rightarrow & a_1 \\ 2 & \leftarrow -- \rightarrow & a_2 \\ 3 & \leftarrow -- \rightarrow & a_3 \\ . & & . \\ . & & . \\ . & & . \\ n & \leftarrow -- \rightarrow & a_n \end{array}$$

When Two infinite sets are equal?

ℕ={1,2,3,4,5...} (natural numbers) (countable)

ℤ={...-2,-1,0,1,2,3,...} (integer numbers)(countable)

ℚ={a/b | a ∈ ℤ and b ∈ ℕ} (rational numbers) (countable)

When there is 1:1 correspondence between them.

$$\begin{array}{ccc} N & 1{:}1 & A \\ 1 & \leftarrow -- \rightarrow & a_1 \\ 2 & \leftarrow -- \rightarrow & a_2 \\ 3 & \leftarrow -- \rightarrow & a_3 \\ . & & . \\ . & & . \\ . & & . \\ n & \leftarrow -- \rightarrow & a_n \end{array}$$

First/Second Year Responses:

Middle of Spring Course:

- A={ℕ} and B={ℤ}
- When the two sets have the same elements.
- Not Answered (32.4%)

End of the Spring Course :

- Two infinite sets are equal if their elements are in 1:1 correspondence with each other.
- They can be placed in one-to-one correspondence.

Senior Colloquium Responses:

End of the Spring Course :

- They have a 1:1 correspondence so they are countable.
- When one element from the first set can be paired with an element from the second set.
- 1:1 correspondence between elements of the sets

Give an example of two infinite sets A and B such that A is a proper subset of B but they have the same number of elements.

ℕ={1,2,3,4,5...} (natural numbers)

ℤ={...-2,-1,0,1,2,3,...} (integer numbers)

ℚ={a/b | a ∈ ℤ and b ∈ ℕ} (rational numbers)

When A= {ℕ} and B={ℚ}, and A= {ℕ} and B= {all even natural numbers}

First/Second Year Responses:

Middle of Spring Course:

- A= { integers divided by 5} and B= { all integers}
- A={x≥0} ⊂ B= {-∞<x<∞}
- A⊂B
- Not Answered (64.8%)

End of the Spring Course :

- Let A = {even numbers} and B = {natural numbers}. A is a proper subset of B, but A and B have the same cardinality.
- A= {All even natural numbers} B={all natural numbers
- A = natural numbers B = integers

Senior Colloquium Responses:

End of Spring Course:

- A={2,4,6,8,10,12,14,16,.......}, B={1,2,3,4,5,6,7,8,...}
- If A is the natural numbers and B is the real numbers, they are both infinite, but A is a subset of B.
- A:{even numbers}, N:{natural numbers}. A<N, 1:1 correspondence of element of A and N

Anna has a set A={all natural numbers divided by 2}
Bob has a set B={all decimals}
Chris has a set C={all even numbers}
Do these sets include each other? How?
Do they have the same number of elements?

3a. A is a subset of C, and C is a subset of B.

3b. A and C are countable.B is not countable.

First/Second Year Responses:

Middle of Spring Course:

- Yes, Set A and C include the same element since even numbers are divisible by 2. Set A and Set C has the same element.
- B is a subset of A while Cis a subset of B and A. No.
- Yes.No.
- Not Answered (part a) 35% part b) 67.5%)

End of the Spring Course :

- Sets A, B, and C do not have the same number of elements. Sets A and C have cardinality Aleph 0, and are in one-to-one correspondence with each other; however, Set B has a cardinality Aleph 1
- They do include each other. even number belong to natural numbers and all decimals belong to natural numbers too. They do not have the same number of elements because decimals have aleph 1 cardinality
- Yes. All even numbers are divisible by 2 and so are decimals such as 8.0 or 4.0. They have the same number of elements because they all go by 2 and all go together.

Senior Colloquium Responses:

End of Spring Course:

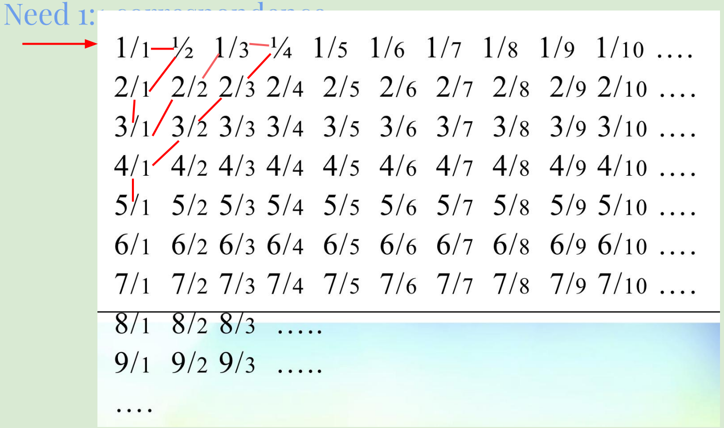
- Set A is a subset of set C, set C is a subset of set B, and A is a subset of B.
- No, B is not in these sets since it does not have whole numbers. A and C are both a subset of each other and all three of these sets are infinite
- These sets include each other as set A is a subset of C and both A and C are subsets of B, furthermore they are all infinite.

Let the infinite set L be the set of points (a,b) on the plane such that a,b are both natural numbers.Can you make an infinite list of all points in set L? Explain how.

We make a list starting at (0,0), then go to (1,0) i.e. one unit to the right (second on the list) , then one unit up (1,1) third on the list, then one unit to the left (0, 1) , then another unit to the left (-1,1), then one unit down, etc. We will visit all points this way and put them on the list.

Example: $\mathbb{N}$={1,2,3,4,5...} and $\mathbb{Q}$={a/b | a $\in \mathbb{Z}$ and b $\in \mathbb{N}$}

Need 1:1 correspondence



First/Second Year Responses:

Middle of Spring Course:

- Yes you can by listing all the decimals (0.000001,0.000002...)
- Yes list the decimal numbers between A and B.
- Yes there is an infinite list of numbers between 2 points (a,b) on a plane.
- Not Answered (64.8%)

End of the Spring Course :

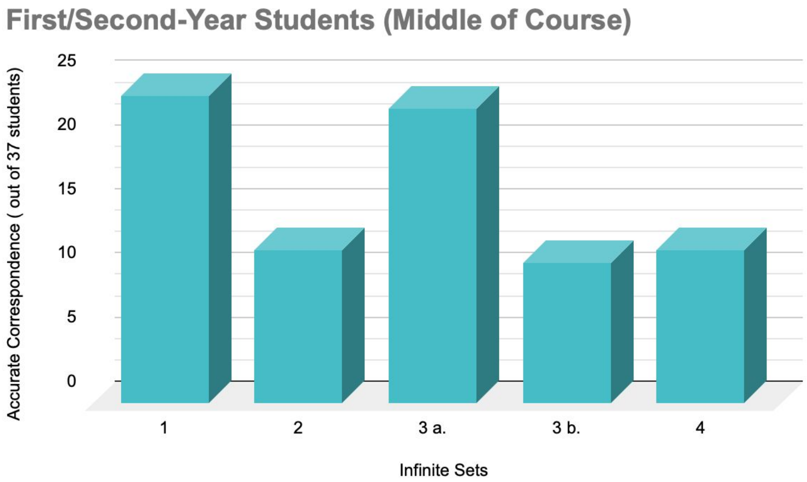
- No, the list is uncountable.
- yes because set L could be on a line where there is different points such as 1.2, 1.5 or 2,3,4.
- Yes, we can. In creating our list since a and b are both natural numbers, we can start with point (1,1), then (2,1), (1,2), (1,3), (2,2), (3,1), (4,1), (3,2), (2,3), (1,4), (1,5), (2,4), (3,3), (4,2), (5,1), etc... In other words, starting from point (1,1), we will visually list the next point to the right of (1,1), then move diagonally up to the point above (1,1), and then move up to the point above (1,2), and then move diagonally down all the way to point (3,1) and continue listing points in a zig-zag manner. In this way, we are able to list all sets of points (a,b) on a plane where a and b are both natural numbers.

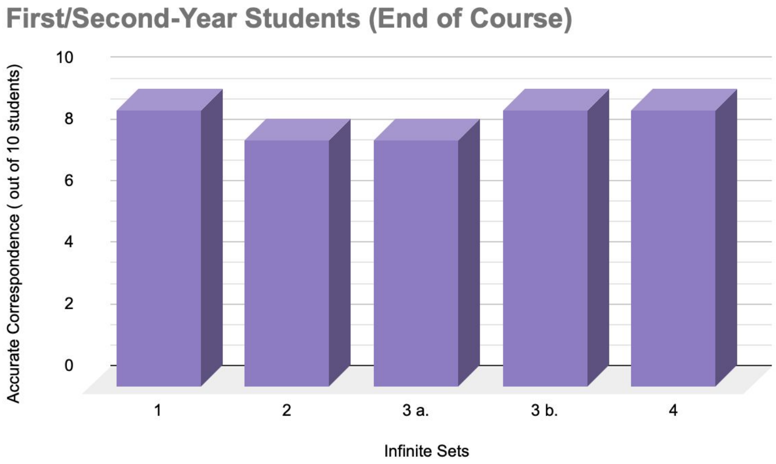
Senior Colloquium Responses:

End of Spring Course:

- Yes, you can make an infinite list of all points in set L. L={(1,1),(1,2),(1,3),(1,4),....,(2,1),(2,2),(2,3),(2,4),....,(3,1),(3,2),....}
- No you can not. If the set infinite, you can not list infinitely many points. You could find some points by taking finite subsets, but we will never be able to list infinitely many points.
- Yes, using one-to-one correspondence. For example, (1,1), (1,2), (1,3)... (2,1), (2,2), (2,3)

First/Second Year Students

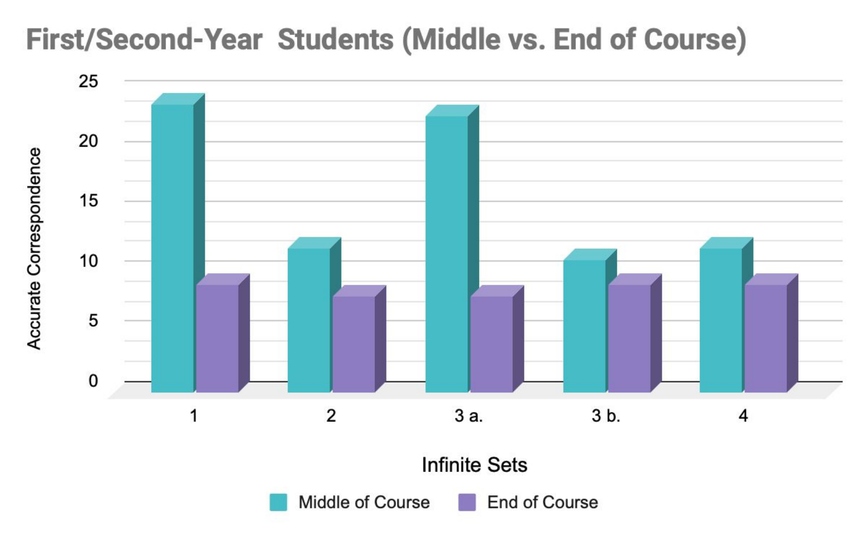




First/Second Year Students

Middle of the Course : Average of 44%

End of the Course: Average of 86%



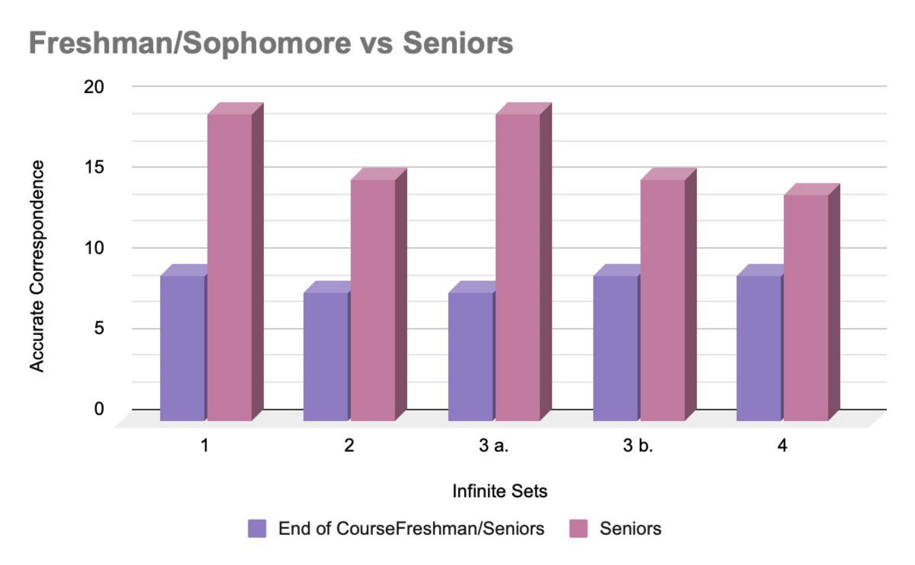
First/Second Year Students vs. Senior Colloquium Students

First/ Second Year students:

End of the Course: Average of 86%

Senior Colloquium Student:

Average of 78%



Bibliography

https://math.libretexts.org/Under_Construction/Stalled_Project_(Not_under_Active_Development)/Additional_Discrete_Topics_(Dean)/Infinite_Sets_and_Cardinality

Lecture Notes Math 230 Logic and Mathematical Reasoning

Dr. Ivona Grzegorczyk

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